Chapter three

Partial Derivatives

Function of two or more variables:

Suppose D is a set of n – tuples of real numbers:

$$(x_1, x_2, \ldots\ldots, x_n)$$

A **real - valued function** f on D is a rule that assigns a unique (single) real number:

$$w = f(x_1, x_2, \ldots\ldots, x_n)$$

To each element in D . The set D is the function's **domain.** The set of w – values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n.

In the function $V = \pi r^2 h$, the dependent variable is V, the independent variables are r and h.

Example: Find value $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point (3,0,4):

***Solution*:**

$$f(3,0,4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{9 + 0 + 16} = \sqrt{25} = 5$$

Limit of a function of two variables:

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_o, y_o), and write:

$$\lim_{(x,y)\to(x_o,y_o)} f(x, y) = L$$

If for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x - x_o)^2 + (y - y_o)^2} < \delta$

Properties of limits of functions of two variables:
The following rules hold if L , M , k are real numbers and:

$$\lim_{(x,y)\to(x_o,y_o)} f(x,y)=L \quad \text{and} \quad \lim_{(x,y)\to(x_o,y_o)} g(x,y)=M$$

1. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y)+g(x,y)) = L+M$

2. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y)-g(x,y)) = L-M$

3. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y).g(x,y)) = L.M$

4. $\lim_{(x,y)\to(x_o,y_o)} (kf(x,y)) = kL$ (any number of k)

5. $\lim_{(x,y)\to(x_o,y_o)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ $M \neq 0$

6. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y))^{r/s} = L^{r/s}$ **(**where r and s are integers and $s \neq 0$ **)**

Example: find the limit of the following:

1. $\lim_{(x,y)\to(0,1)} \dfrac{x-xy+3}{x^2y+5xy-y3}$

2. $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

3. $\lim_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

4. $\lim_{(x,y)\to(1,1)} \dfrac{x^2-2xy+y^2}{x-y}$

5. $\lim_{(x,y)\to(1,1)} \dfrac{x^2-y^2}{x-y}$

***Solution*:**

1. $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2 y + 5xy - y3} = \frac{0-(0)(1)+3}{(0)^2(1)+5(0)(1)-(1)^3} = \frac{3}{0+0-1} = -3$$

2. $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{(3)^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

3. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x^2-xy)}{(\sqrt{x}-\sqrt{y})} . \frac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{x+\sqrt{x}\sqrt{y}-\sqrt{x}\sqrt{y}-y}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0(\sqrt{0}+\sqrt{0}) = 0$$

4. $$\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x-y)^2}{x-y} = \lim_{(x,y)\to(1,1)} (x-y) = 1-1 = 0$$

5. $$\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x+y)\cancel{(x-y)}}{\cancel{(x-y)}}$$

$$= \lim_{(x,y)\to(1,1)} (x+y) = (1+1) = 2$$

Partial Derivatives:

Partial derivatives are the derivatives we get when we hold constant all but one of the independent variable in a function and differentiate with respect to that one.

Partial derivatives of a function of two variables:

The partial derivative of $f(x,y)$ with respect to x at the point (x_o, y_o) is:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_o,y_o)} = \lim_{h\to 0}\frac{f(x_o+h,y_o)-f(x_o,y_o)}{h}$$

Provided the limit exists

The partial derivative of $f(x,y)$ with respect to y at the point (x_o, y_o) is:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_o,y_o)} = \left.\frac{d}{dy}f(x_o,y)\right|_{y=y_o} = \lim_{h\to 0}\frac{f(x_o,y_o+h)-f(x_o,y_o)}{h}$$

Provided the limit exists

$$f_x = \frac{\partial f}{\partial x} \quad , \quad f_y = \frac{\partial f}{\partial y}$$

Example: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if:

$$f(x,y) = x^2 + 3xy + y - 1$$

Solution:

To find $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate with respect to x :

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0 = \boxed{2x + 3y}$$

$\therefore$ The value of $\frac{\partial f}{\partial x}$ at $(4,-5)$ is:

$$= (2)(4) + 3(-5) = 8 - 15 = -7$$

To find $\frac{\partial f}{\partial y}$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3x + 1 - 0 = \boxed{3x + 1}$$

$\therefore$ The value of $\frac{\partial f}{\partial y}$ at $(4,-5)$ is:

$$= (3)(4) + 1 = 12 + 1 = 13$$

Example: Find the values of $\frac{\partial f}{\partial y}$ if $f(x,y) = y \sin xy$

Solution: we treat x as a constant and f as a product of y and $\sin xy$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y)$$

$$= y \cos xy \frac{\partial}{\partial y}(xy) + \sin xy$$

$$= xy \cos xy + \sin xy$$

Example: Find f_x and f_y if $f(x,y)=\dfrac{2y}{y+\cos x}$

Solution:

We treat f as a quotient with y held constant, we get:

$$f_x=\frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial x}(2y)-2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$=\frac{(y+\cos x)(0)-2y(-\sin x)}{(y+\cos x)^2}=\frac{2y\sin x}{(y+\cos x)^2}$$

with x held constant, we get:

$$f_y=\frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial y}(2y)-2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$

$$=\frac{(y+\cos x)(2)-2y(1)}{(y+\cos x)^2}=\frac{2y+2\cos x-2y}{(y+\cos x)^2}$$

$$=\frac{2\cos x}{(y+\cos x)^2}$$

Example: Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ if $f(x,y)=(2x-3y)^3$

Solution:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(2x-3y)^3=3(2x-3y)^2(2)=\boxed{6(2x-3y)^2}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(2x-3y)^3=3(2x-3y)^2(-3)=\boxed{-9(2x-3y)^2}$$

H.W: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the functions:

1. $f(x,y)=(xy-1)^2$　　**2.** $f(x,y)=x^2-xy+y^2$

3. $f(x,y)=(x^2-1)(y+2)$　　**4.** $f(x,y)=\dfrac{x}{(x^2+y^2)}$

functions of more than two variables:

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example: Find $\frac{\partial f}{\partial z}$ if $f(x,y,z)=x\sin(y+3z)$

Solution:

$$\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}\left[x\sin(y+3z)\right]=x\frac{\partial}{\partial z}\sin(y+3z)$$

$$=x\cos(y+3z)\frac{\partial}{\partial z}(y+3z)$$

$$=x\cos(y+3z)(3)=3x\cos(y+3z)$$

Example: Find f_x, f_y and f_z if $f(x,y,z)=xy+yz+xz$

Solution:

$$f_x=\frac{\partial}{\partial x}(xy+yz+xz)=y+z$$

$$f_y=\frac{\partial}{\partial y}(xy+yz+xz)=x+z$$

$$f_z=\frac{\partial}{\partial z}(xy+yz+xz)=y+x$$

***H.W*:** Find f_x, f_y and f_z if $f(x,y,z) = x - \sqrt{y^2 + z^2}$

Second – order partial derivatives:

When we differential a function $f(x,y)$ twice, we produce its second – order derivative. These derivatives are usually denoted by:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Example: If $f(x,y)=x^2y^3+x^4y$, find $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$

Solution:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2y^3+x^4y)=\boxed{2xy^3+4x^3y}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2y^3+x^4y)=\boxed{3x^2y^2+x^4}$$

$\therefore$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}(2xy^3+4x^3y)=2y^3+12x^2y$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}(3x^2y^2+x^4)=6x^2y$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(3x^2y^2+x^4)=6xy^2+4x^3$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}(2xy^3+4x^3y)=6xy^2+4x^3$$

Example: Find f_{xx} , f_{yy} , f_{yx} and f_{xy} if $f(x,y) = x\cos y + ye^x$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \boxed{\cos y + ye^x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = \boxed{-x\sin y + e^x}$$

$\therefore$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + ye^x) = ye^x$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y$$

$$f_{yx} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x\sin y + e^x) = -\sin y + e^x$$

$$f_{xy} = \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y + ye^x) = -\sin y + e^x$$

H.W: Find f_{xx} , f_{yy} , f_{yx} and f_{xy} if $f(x,y) = x + y + xy$

Partial derivatives of higher order:

Although we will deal mostly with first and second – order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third and fourth – order derivatives by symbols like

$$\frac{\partial^3 f}{\partial x\partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2\partial y^2} = f_{yyxx}$$

***The chain Rule for functions of two variables*:**

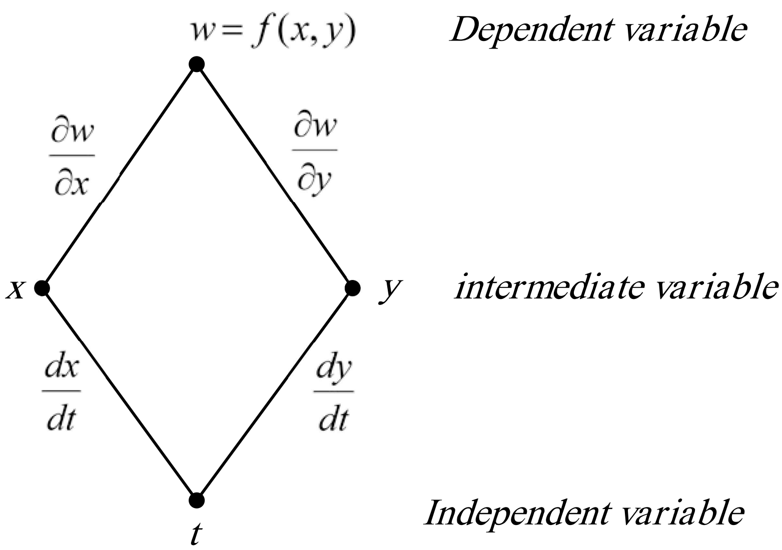
The chain rule formula for function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(y))$ is a differentiable function of t and

$$\frac{df}{dt} = f_x(x(t), y(t)).x'(t) + f_y(x(t), y(t)).y'(t)$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

***Example*:** Use the chain rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. what is the derivatives value at $t = \frac{\pi}{2}$

***Solution*:** we apply the chain rule to find $\frac{dw}{dt}$ as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial(xy)}{\partial x}.\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}.\frac{d}{dt}(\sin t)$$

$$= (y)(-\sin t) + (x)(\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t$$

$$= \cos 2t$$

In this example, we can **check** the result with a more direct calculation. As a function of *t*

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t$$

So

$$\frac{dw}{dt} = \frac{d}{dt}(\frac{1}{2}\sin 2t) = \frac{1}{2}.2\cos 2t = \cos 2t$$

At the given value of *t*:

$$\left(\frac{dw}{dt}\right)_{t=\frac{\pi}{2}} = \cos\left(2.\frac{\pi}{2}\right) = \cos\pi = -1$$

Example: Use the chain rule to find the derivative of $w = x^2 + y^2$ with respect to t $\left(\dfrac{dw}{dt}\right)$, with $x = \cos t$, $y = \sin t$

Solution: we apply the chain rule to find $\dfrac{dw}{dt}$ as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(x^2 + y^2).\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(x^2 + y^2).\frac{d}{dt}(\sin t)$$

$$= (2x)(-\sin t) + (2y)(\cos t)$$

$$= (2\cos t)(-\sin t) + (2\sin t)(\cos t)$$

$$= -2\cos t \sin t + 2\sin t \cos t$$

$$= 0$$

For check:

$$w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

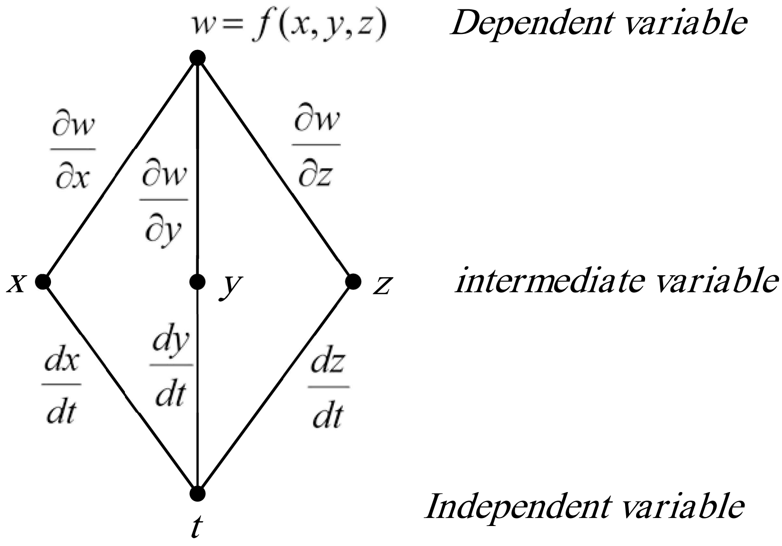
$$\therefore \quad \frac{dw}{dt} = 0$$

H.W: Use the chain rule to find the derivative of $w = x^2 + y^2$ with respect to t $\left(\dfrac{dw}{dt}\right)$, with $x = \cos t + \sin t$, $y = \cos t - \sin t$

***The chain rule for function of three variables*:**

If $w = f(x, y, z)$ is differentiable and x , y and z are differentiable function of t , then w is a differentiable function of t and:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example: Use the chain rule to find the derivative $\left(\frac{dw}{dt}\right)$ of $w = xy + z$ with $x = \cos t$, $y = \sin t$, $z = t$, and determine the value of $\left(\frac{dw}{dt}\right)$ at $t = 0$

Solution:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial}{\partial x}(xy+z).\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(xy+z).\frac{d}{dt}(\sin t) + \frac{\partial}{\partial z}(xy+z).\frac{d}{dt}(t)$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= \cos 2t + 1$$

$$\left(\frac{dw}{dt}\right)_{t=0} = \cos(0) + 1 = 2$$

H.W: Use the chain rule to find the derivative $\left(\frac{dw}{dt}\right)$ of $w = \frac{x}{z} + \frac{y}{z}$ with $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$, and determine the value of $\left(\frac{dw}{dt}\right)$ at $t = 3$

functions defined on surface

If we are interested in the temperature $w = f(x, y, z)$ at points (x, y, z) on a globe in space, we might prefer to think of x , y , and z as functions of the variables r and s that give the points longitudes and latitudes.

If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$ we could then express the temperature as a function of r and s with the composite function.

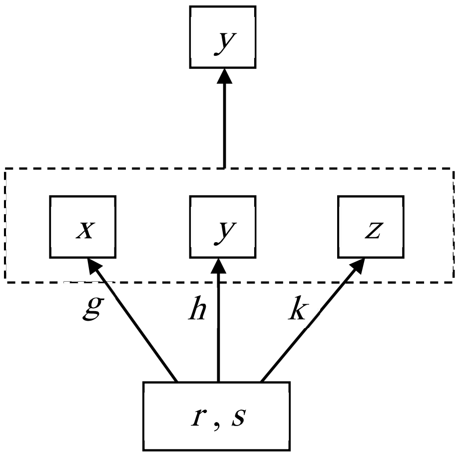
$$w = f(g(r, s), h(r, s), k(r, s))$$

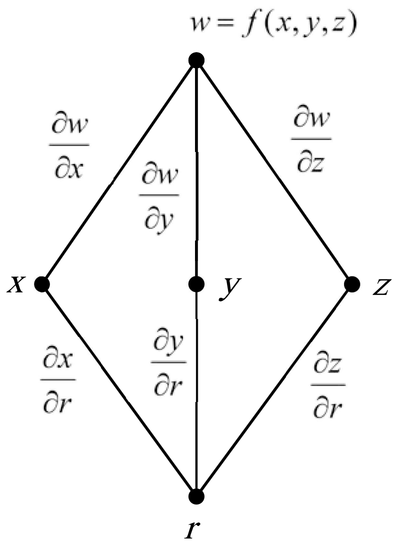
Under the right conditions, w would have partial derivatives with respect to both r and s that could be calculated in the following way:

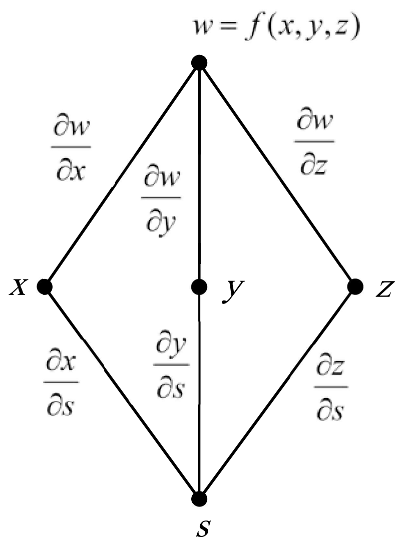
Chain rule for two independent variables and three intermediate variables:
Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$ and $z = k(r, s)$. If all four functions are differentiable, the w has partial derivatives with respect to r and s given by the formulas:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$







$$w = f(g(r, s), h(r, s), k(r, s))$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

If f is a function two variables instead of three, the equations becomes:

If $w = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

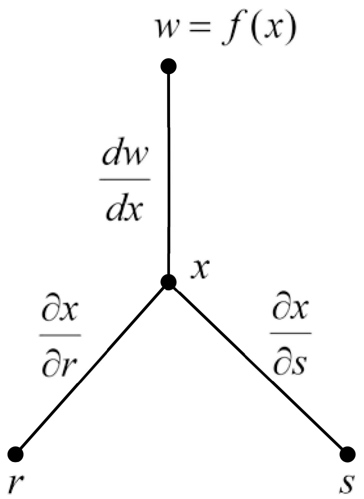
If f is a function of x alone, the equations becomes:

If $w = f(x)$, $x = g(r, s)$, then

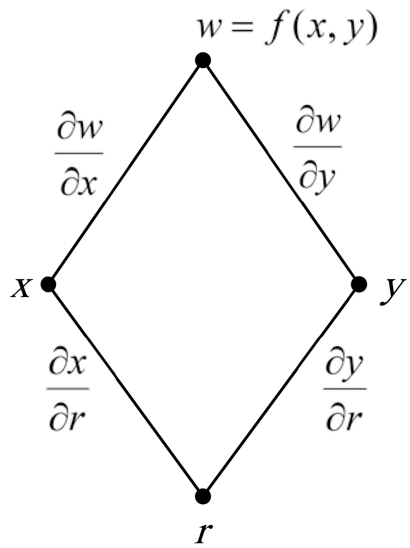
$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$
$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

Example: find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in term of r and s if:

$$w = x + 2y + z^2 \quad , \quad x = \frac{r}{s} \quad , \quad y = r^2 + \ln s \quad , \quad z = 2r$$

Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + (4r)(2)$$

$$= \frac{1}{s} + 4r + 8r$$

$$= \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)(\frac{1}{s}) + (2z)(0)$$

$$= -\frac{r}{s^2} + \frac{2}{s} + 0 = \frac{2}{s} - \frac{r}{s^2}$$

Example: find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in term of r and s if:

$$w = x^2 + y^2 \ , \ x = r - s \quad , \ y = r + s$$

Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$= (2x)(1) + (2y)(1)$$

$$= 2x + 2y$$

$$= 2(r - s) + 2(r + s)$$

$$= 2r - 2s + 2r + 2s$$

$$= 4r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2x)(-1) + (2y)(1)$$

$$= -2x + 2y$$

$$= -2(r - s) + 2(r + s)$$

$$= -2r + 2s + 2r + 2s$$

$$= 4s$$

H.W: find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v if:

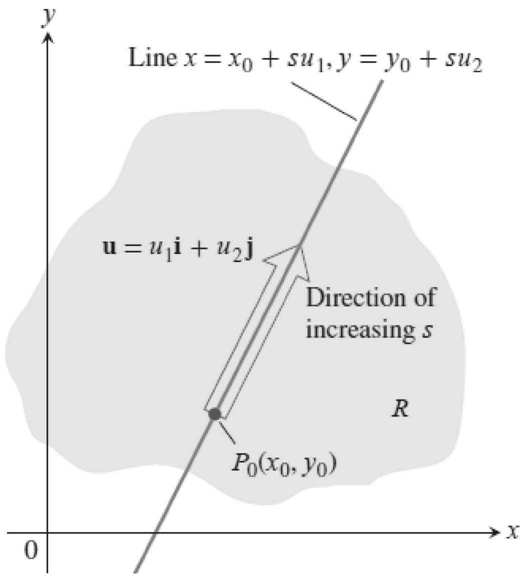
$$z = 4e^x \ln y \ , \ x = \ln(u \cos v) \quad , \ y = u \sin v$$

Directional derivatives and gradient vectors:

Suppose that the function $f(x,y)$ is defined throughout a region R in the xy − plane, that $P_o(x_o, y_o)$ is a point in R and that $u = u_1 i + u_2 j$ is a unit vector. Thus the equations:

$$x = x_o + su_1 \quad , \quad y = y_o + su_2$$

Parameterize the line through P_o parallel to u. if the parameter s measures arc length from P_o in the direction of u , we find the rate of change of f P_o in the direction of u by calculating $\frac{df}{ds}$ at P_o at



The derivative of *f* at $P_o(x_o, y_o)$ in the direction of the unit vector $u = u_1 i + u_2 j$ is the number

$$\left(\frac{df}{ds}\right)_{u,P_o} = \lim_{s\to 0}\frac{f(x_o + su_1, y_o + su_2) - f(x_o, y_o)}{s}$$

The directional derivative is also denoted by:

$(D_u f)_{P_o}$ ⟹ "the derivative of *f* at P_o in the direction of u"

Example: find the derivative of $f(x,y)=x^2+xy$ at $P_o(1,2)$ in the direction of the unit vector $u=\left(\frac{1}{\sqrt{2}}\right)i+\left(\frac{1}{\sqrt{2}}\right)j$

Solution:

$$\left(\frac{df}{ds}\right)_{u,P_o}=\lim_{s\to 0}\frac{f(x_o+su_1,y_o+su_2)-f(x_o,y_o)}{s}$$

$$=\lim_{s\to 0}\frac{f(1+S.\frac{1}{\sqrt{2}},2+s.\frac{1}{\sqrt{2}})-f(1,2)}{s}$$

$$=\lim_{s\to 0}\frac{\left(1+\frac{s}{\sqrt{2}}\right)^2+\left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right)-((1)^2+(1)(2))}{s}$$

$$=\lim_{s\to 0}\frac{\left(1+\frac{2s}{\sqrt{2}}+\frac{s^2}{2}\right)+\left(2+\frac{s}{\sqrt{2}}+\frac{2s}{\sqrt{2}}+\frac{s^2}{2}\right)-(1+2)}{s}$$

$$=\lim_{s\to 0}\frac{\left(1+\frac{2s}{\sqrt{2}}+\frac{s^2}{2}\right)+\left(2+\frac{3s}{\sqrt{2}}+\frac{s^2}{2}\right)-3}{s}$$

$$=\lim_{s\to 0}\frac{\frac{5s}{\sqrt{2}}+\frac{2s^2}{2}}{s}=\lim_{s\to 0}\frac{\frac{5s}{\sqrt{2}}+s^2}{s}=\lim_{s\to 0}\left(\frac{5}{\sqrt{2}}+s\right)=\left(\frac{5}{\sqrt{2}}+0\right)=\frac{5}{\sqrt{2}}$$

The rate of change of $f(x,y)=x^2+xy$ at $P_o(1,2)$ in the direction $u=\left(\frac{1}{\sqrt{2}}\right)i+\left(\frac{1}{\sqrt{2}}\right)j$ is $\frac{5}{\sqrt{2}}$

Gradient vector:

The gradient vector (gradient) of $f(x,y,z)$ at a point $P_o(x_o, y_o, z_o)$ is the vector:

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

obtained by evaluating the partial derivatives of f at P_o

The notation ∇f is read (grad f) as well as (gradient f) and (del f).
The symbol ∇ by itself is read (del)

The directional derivatives

If $f(x,y,z)$ has continuous partial derivatives at $P_o(x_o, y_o, z_o)$ and u is a unit vector, then **the derivative of f at P_o in the direction of u is**:

$$\left(\frac{df}{ds}\right)_{u,P_o} = (\nabla f)_{P_o} . u$$

Which is the scalar product of the gradient of f at P_o and u

Example: find the derivative of $f(x,y,z)=x^3-xy^2-z$ at $P_o(1,1,0)$ in the direction of vector $A=2i-3j+6k$

Solution:

$$u=\frac{A}{|A|}$$

$$|A|=\sqrt{(2)^2+(-3)^2+(6)^2}=\sqrt{4+9+36}=\sqrt{49}=7$$

$$u=\frac{A}{|A|}=\frac{2i-3j+6k}{7}=\frac{2}{7}i-\frac{3}{7}j+\frac{6}{7}k$$

The partial derivatives of f at P_o are

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^3-xy^2-z)=\boxed{3x^2-y^2}$$

$$\therefore\ f_x(1,1,0)=(3)(1)^2-(1)^2=3-1=\boxed{2}$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^3-xy^2-z)=\boxed{-2xy}$$

$$\therefore\ f_y(1,1,0)=-(2)(1)(1)=\boxed{-2}$$

$$f_z=\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(x^3-xy^2-z)=\boxed{-1}$$

$$f_z(1,1,0)=\boxed{-1}$$

The gradient of f at P_o is:

$$\nabla f\big|_{(1,1,0)}=f_x(1,1,0)i+f_y(1,1,0)j+f_z(1,1,0)k=2i-2j-k$$

The derivative of f at P_o in the direction A is therefore:

$$(D_uf)\big|_{(1,1,0)}=\nabla f\big|_{(1,1,,0)}.u$$

$$=(2i-2j-k).\left(\frac{2}{7}i-\frac{3}{7}j+\frac{6}{7}k\right)$$

$$=(2)\left(\frac{2}{7}\right)+(-2)\left(-\frac{3}{7}\right)+(-1)\left(\frac{6}{7}\right)=\frac{4}{7}+\frac{6}{7}-\frac{6}{7}=\boxed{\frac{4}{7}}$$

Example: find the derivative of $f(x,y) = xe^{y} + \cos(xy)$ at the point (2 , 0) in the direction of $v = 3i - 4j$

Solution: the direction of *v* is the unit vector obtained by dividing *v* by its length:

$$u = \frac{v}{|v|}$$

$$|v| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$u = \frac{3i-4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^{y} + \cos(xy)) = \boxed{e^{y} - y\sin(xy)}$$

$$\therefore\ f_x(2,0) = e^0 - (0)\sin(2)(0)) = e^0 - 0 = \boxed{1}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^{y} + \cos(xy)) = \boxed{xe^{y} - x\sin(xy)}$$

$$\therefore\ f_y(2,0) = (2)(e^0) - (2)\sin(2)(0)) = 2e^0 - (2)(0) = (2)(1) - 0 = \boxed{2}$$

The gradient of *f* at (2 , 0) is:

$$\nabla f\big|_{(2,0)} = f_x(2,0)i + f_y(2,0)j = i + 2j$$

The derivative of *f* at (2 , 0) in the direction *v* is therefore:

$$(D_u f)\big|_{(2,0)} = \nabla f\big|_{(2,0)}.u$$

$$= (i+2j).\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

$$= (1)\left(\frac{3}{5}\right) + (2)\left(-\frac{4}{5}\right)$$

$$= \frac{3}{5} - \frac{8}{5} = \frac{-5}{5} = \boxed{-1}$$

Example: find the derivative of function $f(x, y) = 2xy - 3y^2$ at the point P_o (5 , 5) in the direction of $A = 4i + 3j$

Solution:

$$u = \frac{A}{|A|}$$

$$|A| = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$u = \frac{4i + 3j}{5} = \frac{4}{5}i + \frac{3}{5}j$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2xy - 3y^2) = \boxed{2y}$$

$$\therefore\ f_x(5,5) = (2)(5) = \boxed{10}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2xy - 3y^2) = \boxed{2x - 6y}$$

$$\therefore\ f_y(5,5) = (2)(5) - (6)(5) = 10 - 30 = \boxed{-20}$$

The gradient of *f* at (5 , 5) is:

$$\nabla f\big|_{(5,5)} = f_x(5,5)i + f_y(5,5)j = 10i - 20j$$

The derivative of *f* at (5 , 5) in the direction *A* is therefore:

$$(D_u f)\big|_{(5,5)} = \nabla f\big|_{(5,5)} . u$$

$$= (10i - 20j).\left(\frac{4}{5}i + \frac{3}{5}j\right)$$

$$= (10)\left(\frac{4}{5}\right) + (-20)\left(\frac{3}{5}\right)$$

$$= \frac{40}{5} - \frac{60}{5} = \frac{-20}{5} = \boxed{-4}$$

H.W:

1. find the derivative of the function $f(x,y,z) = xy + yz + zx$,at the point $P_o(1,-1,2)$ in the direction of $A = 3i + 6j - 2k$

2. find the derivative of the function $g(x,y,z) = 3e^x \cos yz$,at the point $P_o(0,0,0)$ in the direction of $A = 2i + j - 2k$

Algebra rules for gradients:

1. $\nabla(kf) = k\nabla f$ $\quad$ (*any number k*)

2. $\nabla(f+g) = \nabla f + \nabla g$

3. $\nabla(f-g) = \nabla f - \nabla g$

4. $\nabla(fg) = f\nabla g + g\nabla f$

5. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

***Tangent planes and normal lines*:**

The **tangent plane** at the point $P_o(x_o, y_o, z_o)$ on the level surface $f(x,y,z) = C$ of a differentiable function f is the plane through P_o normal to $\nabla f|_{P_o}$

The **normal line** of the surface at P_o is the line through P_o parallel to $\nabla f|_{P_o}$

The tangent plane and normal line have the following equation:

Tangent plane to $f(x,y,z) = C$ at $P_o(x_o, y_o, z_o)$:

$$f_x(p_o)(x - x_o) + f_y(p_o)(y - y_o) + f_z(p_o)(z - z_o) = 0$$

Normal line to $f(x,y,z) = C$ at $P_o(x_o, y_o, z_o)$:

$$x = x_o + f_x(p_o)t \quad , \quad y = y_o + f_y(p_o)t \quad , \quad z = z_o + f_z(p_o)t$$

Example: find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z-9=0$ at the point $P_o(1,2,4)$

Solution: the tangent plane is:

$$f_x(p_o)(x-x_o)+f_y(p_o)(y-y_o)+f_z(p_o)(z-z_o)=0$$

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z-9)=2x$$

$$f_x(P_o)=f_x(1,2,4)=(2)(1)=\boxed{2}$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z-9)=2y$$

$$f_y(P_o)=f_y(1,2,4)=(2)(2)=\boxed{4}$$

$$f_z=\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(x^2+y^2+z-9)=1$$

$$f_z(P_o)=f_z(1,2,4)=\boxed{1}$$

$\therefore$ The tangent plane is:

$$\boxed{2(x-1)+4(y-2)+(z-4)=0}$$

or

$$2x-2+4y-8+z-4=0$$
$$2x+4y+z-14=0$$
$$2x+4y+z=14$$

The normal line is:

$$x=x_o+f_x(p_o)t\ ,\quad y=y_o+f_y(p_o)t\ ,\quad z=z_o+f_z(p_o)t$$

$$\therefore\ x=1+2t\ ,\qquad y=2+4t\ ,\qquad z=4+t$$

Example: find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z^2=3$ at the point $P_o(1,1,1)$

Solution: the tangent plane is:

$$f_x(p_o)(x-x_o)+f_y(p_o)(y-y_o)+f_z(p_o)(z-z_o)=0$$

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z^2)=2x$$

$$f_x(P_o)=f_x(1,1,1)=(2)(1)=\boxed{2}$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z^2)=2y$$

$$f_y(P_o)=f_y(1,1,1)=(2)(1)=\boxed{2}$$

$$f_z=\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(x^2+y^2+z^2)=2z$$

$$f_z(P_o)=f_z(1,1,1)=(2)(1)=\boxed{2}$$

$\therefore$ The tangent plane is:

$$\boxed{2(x-1)+2(y-1)+2(z-1)=0}$$

or

$2x-2+2y-2+2z-2=0$

$2x+2y+2z-6=0$

$2x+2y+2z=6$

$2(x+y+z)=6 \Longrightarrow x+y+z=3$

The normal line is:

$$x=x_o+f_x(p_o)t\ ,\quad y=y_o+f_y(p_o)t\ ,\quad z=z_o+f_z(p_o)t$$

$$\therefore\quad x=1+2t\ ,\quad y=1+2t\ ,\quad z=1+2t$$

H.W: find the tangent plane and normal line of the surface $f(x,y,z)=x^2+2xy-y^2+z^2=7$ at the point $P_o(1,-1,3)$

Tangent plane to a surface $z = f(x, y)$ at $(x_o, y_o, f(x_o, y_o))$:
The tangent plane to the surface $z = f(x, y)$ of a differentiable function f at the point $P_o(x_o, y_o, z_o) = (x_o, y_o, f(x_o, y_o))$ is:

$$f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o) - (z - z_o) = 0$$

Example: find the tangent plane to the surface $z = x\cos y - ye^x$ at $(0,0,0)$

Solution: the tangent plane

$$f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o) - (z - z_o) = 0$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y - ye^x) = \cos y - ye^x$$

$$f_x(x_o, y_o) = f_x(0,0) = \cos(0) - (0)(e^0) = 1 - (0)(1) = \boxed{1}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y - ye^x) = -x\sin y - e^x$$

$$f_y(x_o, y_o) = f_y(0,0) = -(0)\sin(0) - e^0 = (0)(0) - 1 = \boxed{-1}$$

$\therefore$ The tangent plane is:

$$(1)(x - 0) + (-1)(y - 0) - (z - 0) = 0$$

$$\boxed{(x - 0) - (y - 0) - (z - 0) = 0}$$

or

$$\boxed{x - y - z = 0}$$

Example: find the tangent plane to the surface $z = 4x^2 + y^2$ at point (1,1,5)

Solution: the tangent plane is:

$$f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o) - (z - z_o) = 0$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(4x^2 + y^2) = 8x$$

$$f_x(x_o, y_o) = f_x(1,1) = (8)(1) = \boxed{8}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(4x^2 + y^2) = 2y$$

$$f_y(x_o, y_o) = f_y(1,1) = (2)(1) = \boxed{2}$$

$\therefore$ The tangent plane is:

$$\boxed{8(x-1) + 2(y-1) - (z-5) = 0}$$

or

$8x - 8 + 2y - 2 - z + 5 = 0$

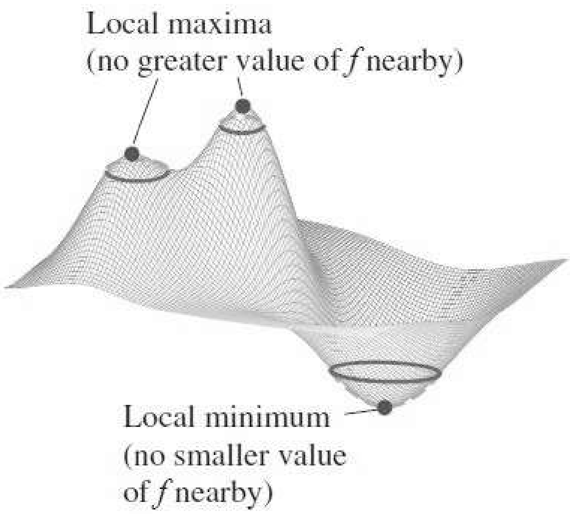
$8x + 2y - z - 5 = 0$

$$\boxed{8x + 2y - z = 5}$$

H.W: find the tangent plane to the surface $z = \sqrt{y - x}$ at the point (1,2,1)

Extreme values and saddle points:

***Derivative test*:**
To find the local extreme values of a function of a single, we look for points where the graph has a horizontal tangent line. At such points, we then look for **local maxima**, **local minima**, and points of inflection. For a function $f(x,y)$ of two variables, we look for points where the surface $z = f(x,y)$ has a horizontal tangent plane. At such points, we then look for **local maxima**, **local minima**, and **saddle points** (more about saddle points in a moment). Local maxima correspond to mountain peak on the surface $z = f(x,y)$ and local minima correspond to valley bottoms. At such points the tangent planes, when they exist are horizontal.
Local extreme are also called ***relative extreme***.



Critical point: an interior point of the domain a function $f(x,y)$ where both f_x and f_y are zero or where one or both f_x and f_y do not exist is *a critical point* of *f*

Saddle point: a differentiable function $f(x,y)$ has a saddle point at critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$ the corresponding point $(a,b,f(a,b))$ on the Surface $z = f(x,y)$ is called *a saddle point* of the surface

***Second derivative test for local extreme values*:**

Suppose that $f(x,y)$ and its first and second partial derivative are continuous throughout a disk centered at (a,b) and that:

$f_x = 0$ and $f_y = 0$ $\Longrightarrow$ solve these equation to find the value of $(x,y) = (a,b)$ $\Longrightarrow$ (*critical point*)

Then:

1. if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) $\Longrightarrow$ then f has **a local maximum** at (a,b)

2. if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) $\Longrightarrow$ then f has **a local minimum** at (a,b)

3. if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b) $\Longrightarrow$ then f has **a saddle point** at (a,b)

4. if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b) $\Longrightarrow$ then **the test inconclusive** at (a,b).
In this case we must find some other way to determine the behavior of f at (a,b)

Example: find the extreme values of the function

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy - x^2 - y^2 - 2x - 2y + 4) = \boxed{y - 2x - 2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy - x^2 - y^2 - 2x - 2y + 4) = \boxed{x - 2y - 2}$$

$$\left.\begin{array}{l} f_x = 0 \Longrightarrow y - 2x - 2 = 0 \\ f_y = 0 \Longrightarrow x - 2y - 2 = 0 \end{array}\right\} \Longrightarrow$$ Solve these equation to find $(x, y) \Longrightarrow (a, b)$

$$\left.\begin{array}{l} x = -2 \Longrightarrow a = -2 \\ y = -2 \Longrightarrow b = -2 \end{array}\right\} \Longrightarrow$$ Critical point $(-2,-2)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(y - 2x - 2) = -2$$

$$\therefore\ f_{xx}(-2,-2) = \boxed{-2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x - 2y - 2) = -2$$

$$\therefore\ f_{yy}(-2,-2) = \boxed{-2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(y - 2x - 2) = 1$$

$$\therefore\ f_{xy}(-2,-2) = \boxed{1}$$

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-2)(-2) - (1)^2 = 4 - 1 = \boxed{3}$$

$f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0 \Longrightarrow \therefore f$ has a local maximum at $(-2,-2)$

The value of f at this point is:

$$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - (2)(-2) - (2)(-2) + 4$$
$$= 4 - 4 - 4 + 4 + 4 + 4 = \boxed{8}$$

Example: find the local maxima, local minima, and saddle point of the function

$f(x,y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$

Solution:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + 3y^2 - 6x + 3y - 6) = \boxed{2x + 3y - 6}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + 3y^2 - 6x + 3y - 6) = \boxed{3x + 6y + 3}$$

$f_x = 0 \Longrightarrow 2x + 3y - 6 = 0$

$f_y = 0 \Longrightarrow 3x + 6y + 3 = 0$

$\Longrightarrow$ Solve these equation to find $(x, y) \Longrightarrow (a, b)$

$x = 15 \Longrightarrow a = 15$

$y = -8 \Longrightarrow b = -8$

$\Longrightarrow$ Critical point $(15,-8)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 3y - 6) = 2$$

$\therefore\ f_{xx}(15,-8) = \boxed{2}$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x + 6y + 3) = 6$$

$\therefore\ f_{yy}(15,-8) = \boxed{6}$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 3y - 6) = 3$$

$\therefore\ f_{xy}(15,-8) = \boxed{3}$

$f_{xx} f_{yy} - f_{xy}{}^2 = (2)(6) - (3)^2 = 12 - 9 = \boxed{3}$

$f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}{}^2 > 0 \Longrightarrow \therefore f$ has a local minimum at $(15,-8)$

The value of f at this point is:

$$f(15,-8) = (15)^2 + (3(15)(-8) + (3)(-8)^2 - (6)(15) + (3)(-8) - 6$$
$$= 225 - 360 + 192 - 90 - 24 - 6 = \boxed{-63}$$

H.W: find the local maxima, local minima, and saddle point of the functions:

1. $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

2. $f(x,y) = x^2 + xy + 3x + 2y + 5$

3. $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

4. $f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$

Lagrange multipliers:

Sometimes we need to find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane – a disk, for example a closed triangular region, or along a curve. The method of Lagrange multipliers is a powerful method for finding extreme values of constrained function.

The method of Lagrange multipliers:

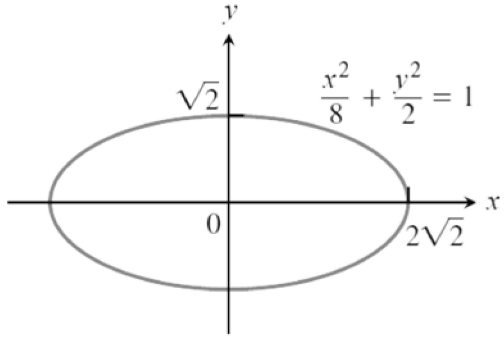
Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable. To find the local maximum and minimum values of *f* subject to the constraint $g(x,y,z) = 0$, find the values of x, y, z and λ that simultaneously satisfy the equations:

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 0$$

Where λ (Lagrange multiplier)

For function of two independent variables, the condition is similar, but without the variable *z*.

Example: find the greatest and smallest values that the function $f(x,y)=xy$ takes on the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$



Solution: we want the extreme values of $f(x,y)=xy$ subject to the constraint $g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}-1=0$, to do so, we first find the value of x , y , and λ for which:

$$\nabla f=\lambda\nabla g \quad \text{and} \quad g(x,y)=0$$

$$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j$$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(xy)=y$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(xy)=x$$

$$\therefore \quad \nabla f=yi+xj$$

$$\nabla g=\frac{\partial g}{\partial x}i+\frac{\partial g}{\partial y}j$$

$$\frac{\partial g}{\partial x}=\frac{\partial}{\partial x}(\frac{x^2}{8}+\frac{y^2}{2}-1)=\frac{2x}{8}=\frac{x}{4}$$

$$\frac{\partial g}{\partial y}=\frac{\partial}{\partial y}(\frac{x^2}{8}+\frac{y^2}{2}-1)=\frac{2y}{2}=y$$

$$\therefore \quad \nabla g=\frac{x}{4}i+yj$$

$\therefore$ the gradient equation gives:

$$yi + xj = \lambda(\frac{x}{4}i + yj)$$

$$yi + xj = \frac{\lambda}{4}xi + \lambda yj$$

From which we find

$$y = \frac{\lambda}{4}x \qquad , \qquad x = \lambda y$$

$$\therefore\ y = \frac{\lambda}{4}(\lambda y) \Longrightarrow y = \frac{\lambda^2}{4}y$$

So that $y = 0$ or $\lambda = \pm 2$

We now consider these two cases:

Case1: If $y = 0$, then $x = y = 0$, but the point $(0,0)$ is not on the ellipse, hence $y \neq 0$

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$

Substituting this in the equation $g(x, y) = 0$ gives:

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1$$

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1 \Longrightarrow 4y^2 + 4y^2 = 8 \Longrightarrow 8y^2 = 8 \Longrightarrow y^2 = 1 \Longrightarrow y = \pm 1$$

The function $f(x, y) = xy$, therefore takes on its extreme values on the ellipse at the four point $(\pm 2,1), (\pm 2,-1)$.
The extreme values are $xy = 2$ and $xy = -2$

Example: find the maximum and minimum values of the function $f(x,y)=3x+4y$ on the circle $x^2+y^2=1$

Solution: we model this as a Lagrange multiplier problem with
$f(x,y)=3x+4y \quad , \quad g(x,y)=x^2+y^2-1$

and look for the values of x , y , and λ that satisfy the equations:

$$\nabla f=\lambda\nabla g$$

$$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j$$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(3x+4y)=3$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(3x+4y)=4$$

$$\therefore \quad \nabla f=3i+4j$$

$$\nabla g=\frac{\partial g}{\partial x}i+\frac{\partial g}{\partial y}j$$

$$\frac{\partial g}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2-1)=2x$$

$$\frac{\partial g}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2-1)=2y$$

$$\therefore \quad \nabla g=2xi+2yj$$

$$\therefore \qquad 3i+4j=\lambda(2xi+2yj)$$
$$3i+4j=2\lambda xi+2\lambda yj$$

$$g(x,y)=0 \quad \Longrightarrow \quad x^2+y^2-1=0$$

$3 = 2\lambda x \quad \Longrightarrow \quad x = \dfrac{3}{2\lambda}$

$4 = 2\lambda y \quad \Longrightarrow \quad y = \dfrac{4}{2\lambda} = \dfrac{2}{\lambda}$

$$\therefore \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0$$

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1 \Longrightarrow 9 + 16 = 4\lambda^2 \Longrightarrow 25 = 4\lambda^2 \Longrightarrow \lambda^2 = \frac{25}{4} \Longrightarrow \lambda = \pm\frac{5}{2}$$

Thus:

$$x = \frac{3}{2\lambda} = \pm\frac{3}{(2)\left(\frac{5}{2}\right)} = \pm\frac{3}{5}$$

$$y = \frac{2}{\lambda} = \pm\frac{2}{\left(\frac{5}{2}\right)} = \pm\frac{4}{5}$$

And $f(x, y) = 3x + 4y$ has extreme values at the points:

$$(x, y) = \pm\left(\frac{3}{5}, \frac{4}{5}\right)$$

By calculating the value of $3x + 4y$ at the points $\pm\left(\frac{3}{5}, \frac{4}{5}\right)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are :

$$(3)\left(\frac{3}{5}\right) + (4)\left(\frac{4}{5}\right) = \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5$$

And

$$(3)\left(-\frac{3}{5}\right) + (4)\left(-\frac{4}{5}\right) = -\frac{9}{5} - \frac{16}{5} = -\frac{25}{5} = -5$$

***Lagrange multipliers with two constraints*:**
Many problems require us to find the extreme values of a differentiable function $f(x,y,z)$ whose variables are subject to two constraints. If the constraints are :

$$g_1(x,y,z)=0 \quad \text{and} \quad g_2(x,y,z)=0$$

g_1 and g_2 are differentiable with ∇g_1 not parallel to ∇g_2 , we find the constrained local maxima and minima of *f* by introducing two Lagrange multipliers λ and μ.
That is, we locate the points $P(x,y,z)$ where *f* takes on its constrained extreme values by finding the values of x , y, λ and μ that simultaneously satisfy the equations

$$\nabla f=\lambda\nabla g_1+\mu\nabla g_2$$
$$g_1(x,y,z)=0 \qquad g_2(x,y,z)=0$$

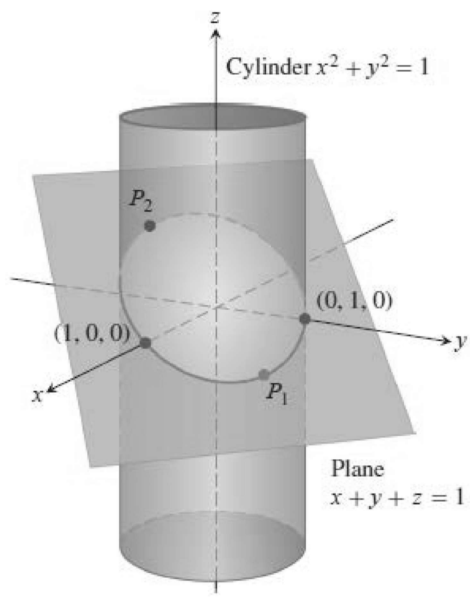
***Example*:** the plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin:

***Solution*:**
We find the extreme values of

$f(x,y,z)=x^2+y^2+z^2$

$g_1(x,y,z)=x^2+y^2-1=0$ ……..(1)

$g_2(x,y,z)=x+y+z-1=0$ ……..(2)



$$\nabla f=\lambda\nabla g_1+\mu\nabla g_2$$

$$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z^2)=2x$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z^2)=2y$$

$$\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(x^2+y^2+z^2)=2z$$

$$\therefore \quad \nabla f = 2xi + 2yj + 2zk$$

$$\nabla g_1 = \frac{\nabla g_1}{\partial x} i + \frac{\nabla g_1}{\partial y} j + \frac{\nabla g_1}{\partial z} k$$

$$\frac{\partial g_1}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 - 1) = 2x$$

$$\frac{\partial g_1}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 - 1) = 2y$$

$$\frac{\partial g_1}{\partial z} = \frac{\partial}{\partial z}(x^2 + y^2 - 1) = 0$$

$$\therefore \quad \nabla g_1 = 2xi + 2yj$$

$$\nabla g_2 = \frac{\nabla g_2}{\partial x} i + \frac{\nabla g_2}{\partial y} j + \frac{\nabla g_2}{\partial z} k$$

$$\frac{\partial g_2}{\partial x} = \frac{\partial}{\partial x}(x + y + z - 1) = 1$$

$$\frac{\partial g_2}{\partial y} = \frac{\partial}{\partial y}(x + y + z - 1) = 1$$

$$\frac{\partial g_2}{\partial z} = \frac{\partial}{\partial z}(x + y + z - 1) = 1$$

$$\therefore \quad \nabla g_2 = i + j + k$$

$$\therefore \quad 2xi + 2yj + 2zk = \lambda(2xi + 2yj) + \mu(i + j + k)$$

$$2xi + 2yj + 2zk = (2\lambda x + \mu)i + (2\lambda y + \mu)j + \mu k$$

$$\left.\begin{aligned} 2x &= 2\lambda x + \mu \\ 2y &= 2\lambda y + \mu \\ 2z &= \mu \end{aligned}\right\} \text{........(3)}$$

$$\left.\begin{aligned} 2x = 2\lambda x + 2z \;\Longrightarrow\; 2z = 2x - 2\lambda x \;\Longrightarrow\; z = x(1-\lambda) \\ 2y = 2\lambda y + 2z \;\Longrightarrow\; 2z = 2y - 2\lambda y \;\Longrightarrow\; z = y(1-\lambda) \end{aligned}\right\} \text{.....(4)}$$

Equation (4) are satisfied simultaneously if either

$\lambda = 1$ and $z = 0$

or

$\lambda \neq 1$ and $x = y = \dfrac{z}{(1-\lambda)}$

If $z = 0$, then solving equations (1) and (2) simultaneously to find the corresponding points on the ellipse gives the two points $(1,0,0)$ and $(0,1,0)$

If $x = y$, then equation (1) and (2) give:

$$x^2 + y^2 - 1 = 0 \;\Longrightarrow\; 2x^2 = 1 \;\Longrightarrow\; x^2 = \frac{1}{2} \;\Longrightarrow\; \therefore\; x = \pm\frac{1}{\sqrt{2}}$$

$$x + y + z = 1 \;\Longrightarrow\; z = 1 - 2x \;\Longrightarrow\; z = 1 \pm \frac{2}{\sqrt{2}} \;\Longrightarrow\; z = 1 \pm \sqrt{2}$$

$\therefore$ the corresponding points on the ellipse are:

$$P_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 - \sqrt{2}\right)$$

and

$$P_2 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1 + \sqrt{2}\right)$$

Have we need to be careful, however. Although P_1 and P_2 both give local maxima of *f* on the ellipse, P_2 is farther from the origin than P_1 .
The points on the ellipse closest to the origin are $(1,0,0)$ and $(0,1,0)$.
The point on the ellipse farthest from the origin is P_2